

ON SINGULAR INTEGRAL EQUATIONS FOR KINKED CRACKS

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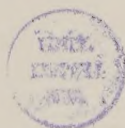
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Abstract

Chebyshev polynomial techniques for solution of singular integral equations leading to square root singularities at the ends of the interval of integration are studied. It is shown that the results are less accurate when a singularity, albeit a weak one, appears between the interval ends. Typical examples are problems involving kinked cracks. Some attempts to improve the accuracy are discussed.

1. Introduction

A commonly practised method to solve singular integral equations leading to square root singularities at the ends of the interval of integration is to use series of Chebyshev polynomials [1]. Since the appropriate quadrature formulas employ sampling points that cannot be arbitrarily placed in the interval of integration, difficulties should be expected when a singularity, even a weak one, appears between the interval ends. Typical examples are problems regarding kinked cracks. The accuracy of solutions employing Chebyshev quadratures to such problems is investigated. Comparison is made with similar problems without singularities between the ends of the integration interval.



2. The equation

The formulation of plane linearly elastic problems involving a crack $(x, y(x))$, $a_0 \leq x \leq d_0$, described as a continuous distribution of dislocations results in two coupled singular integral equations [2,3]. After a change of coordinates in order to describe the crack in parametric form, $(x(t), y(t))$, $-1 \leq t \leq 1$, and separation of the singular part, the equations take the form, cf. [2]

$$\sum_{k=x}^y \left[c_{lk}(t) \int_{-1}^1 b_k(\tau) (t-\tau)^{-1} d\tau + \int_{-1}^1 R_{lk}(t, \tau) b_k(\tau) d\tau \right] \\ = \left[[x'(t)]^2 + [y'(t)]^2 \right] \sigma_l(t) = f_l(t) \quad -1 \leq t \leq 1 \quad (1)$$

where

$l=n, t$ (n =normal, t =tangential)

$$b_k(t) = 2G[\pi(\kappa+1)]^{-1} du_k/dt$$

$u_k(t)$ are the unknown displacements

$$\kappa = \begin{cases} (3-\nu)/(1+\nu) & \text{in plane stress} \\ 3-4\nu & \text{in plane strain} \end{cases}$$

ν = Poisson's ratio

and c_{lk} , $R_{lk}(t, \tau)$ and $f_l(t)$ are known non-singular functions over the interval $-1 \leq t \leq 1$. It is assumed that $y(x)$ is a single-valued function of x and that the solid is infinite in the plane. The first integral in eq. (1) is to be taken in the sense of the Cauchy principal value.

The accuracy of a numerical solution to (1) with the technique using Chebyshev polynomials [1] will be investigated.

3. Solution by Chebyshev polynomials

Since the unknown functions $b_k(t)$ have square-root singularities at the ends of the interval $-1 \leq t \leq 1$, it is assumed that they can be represented by the expansion

$$b_k(t) \approx (1-t^2)^{-1/2} \sum_{j=0}^N B_{kj} T_j(t) = (1-t^2)^{-1/2} b_k^{(0)}(t) \quad -1 \leq t \leq 1 \quad (2)$$

$T_j(t)$ are Chebyshev polynomials of the first kind with the associated weight function $(1-t^2)^{-1/2}$, and B_{kj} are constants.

The stress intensity factors at the crack-tip $x = d_0$, i.e. $t = 1$, are given by the limiting values

$$K_{I, II} \approx \pi [2\pi / (d_0 - a_0)]^{-1/2} \lim_{t \rightarrow 1} \{ b_{n,t}^{(0)}(t) [1 + [y'(t)/x'(t)]^2]^{-1/2} \} \quad (3)$$

where

$$b_{n,t}^{(0)} = [b_{y,x}^{(0)} x'(t) \mp b_{x,y}^{(0)} y'(t)] [x'(t)^2 + y'(t)^2]^{-1/2}$$

and upper sign goes with first index. Since $T_j(1) = 1$, use of eq. (2) shows that the expressions for K_I and K_{II} can be written:

$$K_{I, II} \approx \pi [2\pi / (d_0 - a_0)]^{1/2} [x'(1)]^{1/2} [x'(1)^2 + y'(1)^2]^{-3/4} \times \sum_{j=0}^N B_{(n,t)j}$$

where

$$B_{(n,t)j} = B_{(y,x)j} x'(1) \mp B_{(x,y)j} y'(1) \quad (4)$$

The condition that the crack closes at the ends is

$$\int_{-1}^1 b_k(t) dt = 0 \quad (5)$$

Insertion of (2) into (5) and use of the orthogonality relation [4]

$$\int_{-1}^1 T_i(t) T_j(t) (1-t^2)^{-1/2} dt = \begin{cases} 0 & i \neq j \\ \pi & i=j=0 \\ \pi/2 & i=j>0 \end{cases} \quad (6)$$

gives $B_{k0} = 0$.

Substitution of (2) into (1) gives two coupled approximate relations

$$\sum_{k=x}^y \left\{ \sum_{j=1}^N B_{kj} \left[-\pi c_{1k}(t) U_{j-1}(t) + \int_{-1}^1 R_{1k}(t, \tau) T_j(\tau) (1-\tau^2)^{-1/2} d\tau \right] \right\} \approx f(t) \quad (7)$$

where the relation [5]

$$\int_{-1}^1 T_i(\tau) (\tau-t)^{-1} (1-\tau^2)^{-1/2} d\tau = \begin{cases} 0 & i=0 \\ \pi U_{i-1}(t) & i>0 \end{cases} \quad (8)$$

has been used. The integral is to be taken in the sense of the Cauchy principal value.

Quadrature formulas for Chebyshev polynomials will now be used. The first one reads, see e.g. [4]:

$$\int_{-1}^1 f(t) (1-t^2)^{-1/2} dt \approx \sum_{q=1}^Q w_q f(t_q) \quad (9)$$

$$w_q = \pi/Q \quad \text{and} \quad t_q = \cos[(2q-1)\pi/2Q]$$

The related polynomial is the Chebyshev polynomial of the first kind, $T_q(t) = T_q(\cos\theta) = \cos(q\theta)$.

The second quadrature formula reads [4]:

$$\int_{-1}^1 f(t) (1-t^2)^{1/2} dt \approx \sum_{p=1}^P w_p f(t_p) \quad (10)$$

$$w_p = \pi(\sin\theta_p)^2/(P+1), \quad \theta_p = p\pi/(P+1) \quad \text{and} \quad t_p = \cos\theta_p$$

The related polynomial is the Chebyshev polynomial of the second kind, $U_p(t) = U_p(\cos\theta) = \sin[(p+1)\theta]/\sin\theta$.

By means of quadrature formula (9) the relation (7) can be written

$$\sum_{k=x}^y \left\{ \sum_{j=1}^N B_{kj} [-\pi c_{kl}(t) U_{j-1}(t) + \frac{\pi}{Q} \sum_{q=1}^Q R_{lk}(t, \tau_q) T_j(\tau_q)] \right\} \approx f_1(t) \quad (11)$$

$$\tau_q = \cos[(2q-1)\pi/2Q]$$

Multiplication of both sides of equations (11) by $U_i(t)(1-t^2)^{1/2}$, $i = 0, 1, \dots, N-1$, and integration over the interval $-1 \leq t \leq 1$ gives a system of $2N$ linear algebraic equations in the $2N$ unknown coefficients B_{kj} . Use of quadrature formula (10) gives

$$\begin{aligned} \sum_{k=x}^y \left\{ \sum_{j=1}^N B_{kj} \left[-\pi \sum_{p=1}^P c_{lk}(t_p) U_{j-1}(t_p) U_i(t_p) \sin^2 \theta_p \right. \right. \\ \left. \left. + \frac{\pi}{Q} \sum_{p=1}^P \sum_{q=1}^Q R_{lk}(t_p, \tau_q) T_j(\tau_q) U_i(t_p) \sin^2 \theta_p \right] \right\} \\ \approx \sum_{p=1}^P f_1(t_p) U_i(t_p) \sin^2 \theta_p \quad i=0, 1, 2, \dots, N-1 \end{aligned} \quad (12)$$

$$\begin{aligned} \text{where} \quad t_p &= \cos \theta_p & \theta_p &= p\pi/(P+1) \\ \tau_q &= \cos[(2q-1)\pi/2Q] \end{aligned}$$

4. The circular arc crack

In order to investigate the accuracy of the equation system (12), the stress intensity factors K_I and K_{II} were calculated for a circular arc crack with opening angle 2α , subjected to remote stresses σ_y , σ_x and τ_{xy} according to Fig. 1. Comparison can be made with the analytical expressions, which read [6]:

$$K_I/\sqrt{\pi a} = \left[\frac{1}{2}(\sigma_y + \sigma_x) - \frac{1}{2}(\sigma_y - \sigma_x) \left(\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \right)^2 \right] \cos \frac{\alpha}{2} [1 + \left(\sin \frac{\alpha}{2} \right)^2]^{-1} \\ + \frac{1}{2}(\sigma_y - \sigma_x) \cos \frac{3\alpha}{2} - \tau_{xy} \left(\sin \frac{3\alpha}{2} + \left(\sin \frac{\alpha}{2} \right)^3 \right) \quad (13)$$

$$K_{II}/\sqrt{\pi a} = \left[\frac{1}{2}(\sigma_y + \sigma_x) - \frac{1}{2}(\sigma_y - \sigma_x) \left(\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \right)^2 \right] \sin \frac{\alpha}{2} [1 + \left(\sin \frac{\alpha}{2} \right)^2]^{-1} \\ + \frac{1}{2}(\sigma_y - \sigma_x) \sin \frac{3\alpha}{2} + \tau_{xy} \left[\cos \frac{3\alpha}{2} + \cos \frac{\alpha}{2} \left(\sin \frac{\alpha}{2} \right)^2 \right] \quad (14)$$

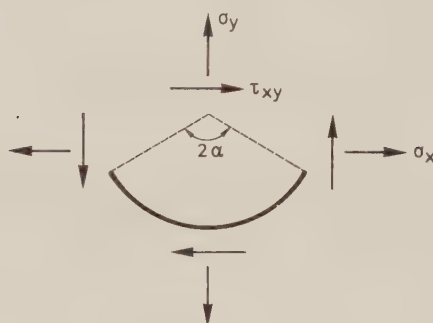


Fig. 1. The circular arc crack

The error in the stress intensity factors K_I and K_{II} from calculations according to equations (12) with the choices $N = 20$ and $P = Q = 50$ increases with increasing angle α but is still at $\alpha = 4\pi/9$ only about one per cent.

Since slight oscillations in the solutions $b_k^{(o)}(t)$ depending on the choice of N were observed, an attempt was made to increase the accuracy by satisfying relations (11) by means of the method of least squares. To this end the two relations (11) were assumed to be satisfied at the points $t = t_m = \cos[m\pi/(M+1)]$, $m = 1, 2, \dots, M$, where $M > N$, still approximately. Reduction of the resulting system of $2M$ equations to a system with $2N$ equations was achieved by forming the sum of the squares of the differences between the right and left members and then minimizing this sum with respect to the $2N$ unknown coefficients B_{kj} . This results in the desired system of $2N$ equations.

This method actually proved to increase the accuracy significantly. With the

choice $N = 20$, $M = 4N$ and $Q = 50$, K_I and K_{II} could be calculated to within 0.05 percent from the theoretical value at $\alpha = 4\pi/9$.

A way to further improve the accuracy of the solution to system (12) in the vicinity of the crack tip at $t = 1$, is to perform the calculations in two steps. A first solution $b_k^{(1)}(t)$ is obtained according to (12). The error in this solution is anticipated to be very small except in the neighbourhood of the crack tip at $t = 1$, say between $t = t_a$ and $t = 1$, where t_a is close to unity. After introduction of a new coordinate $\eta = (2t-1-t_a)/(1-t_a)$ in the interval $t_a < t \leq 1$, a second solution may be written

$$\begin{aligned} b_k^{(2)}(t) &= b_k^{(1)}(t) & \text{for } -1 \leq t \leq t_a \\ b_k^{(2)}(t) &= b_k^{(1)}(t) + c_k(\eta) & \text{for } t_a < t \leq 1 \end{aligned} \quad (15)$$

where

$$c_k(\eta) \approx (1-\eta^2)^{-1/2} \sum_{j=0}^N C_{kj} T_j(\eta) \quad -1 < \eta \leq 1 \quad (16)$$

Condition (5) and use of (6) gives $C_{k0} = 0$.

Insertion of (15) into (1) gives two equations for $c_k(\eta)$ since $b_k^{(1)}(t)$ is known. These equations are solved in the same manner as previously described.

With this method of solution, using the value $t_a = 0.8$ and making the choices $N = 20$ and $P = Q = 50$ in both steps, the stress intensity factors are calculated with the same accuracy as with the method of least squares, i.e. the error is about 0.05 per cent for $\alpha = 4\pi/9$.

The approach with a solution in two steps appears to be suitable in cases where the crack configuration changes substantially near one end, since the sampling points (t_p, τ_q) are fixed when P and Q are chosen, i.e. it is not possible to increase the density of sampling points at special portions of the interval of integration.

4. The V-shaped crack

To investigate the influence on the solution by a sharp corner, a circular arc crack, successively deformed towards the shape of a V is studied, see Fig. 2.

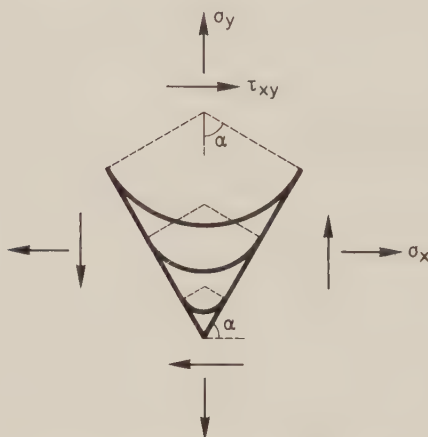


Fig. 2. An arc-shaped crack successively deformed towards a V-shape.

As the V-shape develops the nature of the solution changes. The functions $b_k^{(o)}(t)$ show increasing oscillations in the whole interval $-1 < t < 1$ as the corner develops, and it is obvious that the singular nature of the corner influences the solution. The magnitudes of the coefficients B_{kj} alter from that of a rapid decrease with increasing order j for a circular arc crack to an increase with j for the V-shape. The solutions $b_n^{(o)}(t)$ and $b_t^{(o)}(t)$ are shown in Fig. 3 for the two cases with a circular crack and a V-shaped crack. The coefficients B_{kj} calculated after making the choice $N = 20$ and $P = Q = 50$ are shown in Fig. 4.

Use of the method of least squares is not sufficient to stabilize the solution for the V-shape. The stress intensity factors K_I and K_{II} vary substantially with N , P and Q and, judging from this variation, it is hardly possible to obtain an accuracy on the one per cent level as compared to the 0.05 per cent level easily obtained for the circular crack.

The example was chosen because the analytical solution is known for a circular crack. Of more practical interest, however, is the case of a straight crack with a small kink.

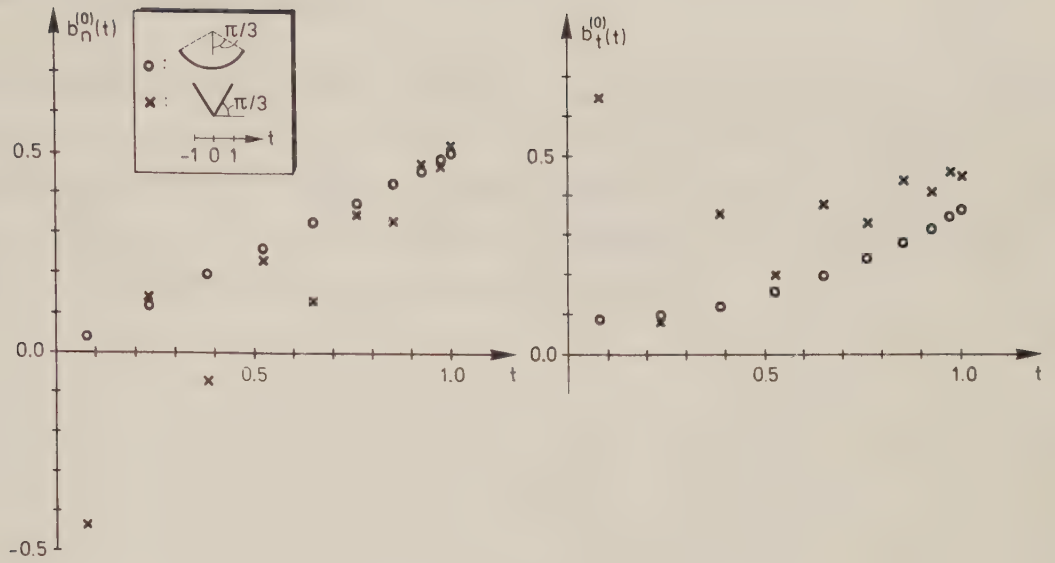


Fig. 3. Values of $b_n^{(o)}(t)$ (symmetrical) and $b_t^{(o)}(t)$ (antisymmetrical) for an arc-shaped and a V-shaped crack at points $t = t_r = (2r-1)/2N$, $r = 1, 2, \dots, N$. $N = 20$, $P = Q = 50$. $\sigma_y/\sigma_x = 2$, $\tau_{xy} = 0$.

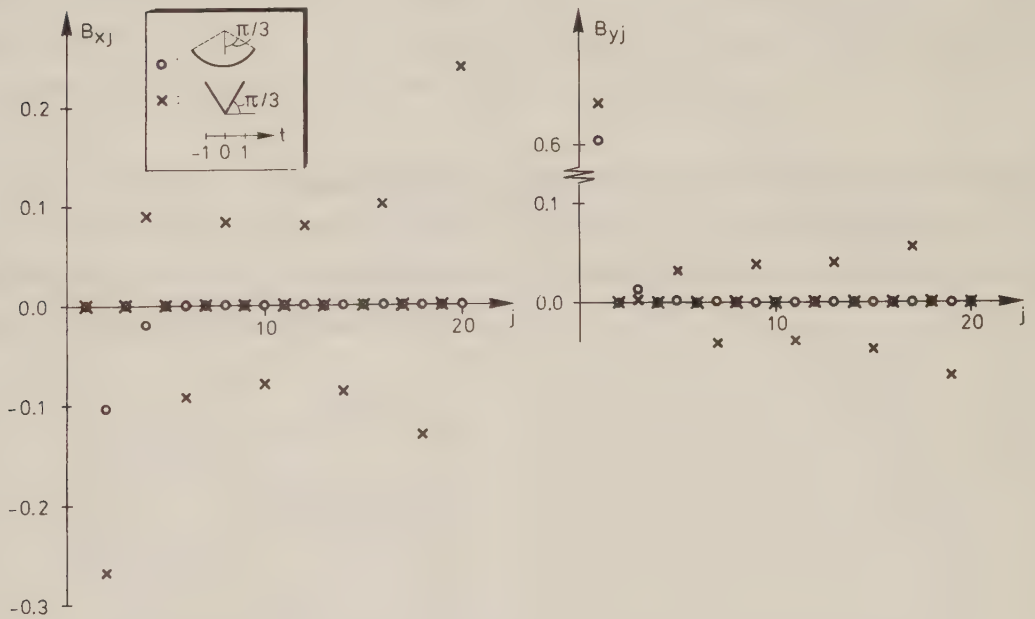


Fig. 4. Magnitude of coefficients B_{kj} , $j = 1, 2, \dots, N$ for an arc-shaped and a V-shaped crack. $N = 20$, $P = Q = 50$. $\sigma_x/\sigma_y = 2$, $\tau_{xy} = 0$.

5. The smoothly curved crack

As a preliminary study of the solution for a straight crack with a small kink, a smoothly curved crack according to Fig. 5 is considered. The crack is straight between $x = a_0$ and $x = c_0$. The parameter t equals -1 for $x = a_0$ and $+1$ for $x = d_0$. For $x = c_0$ the parameter t is denoted t_c . The crack direction at the tip $x = d_0$ forms the angle α to the straight crack.

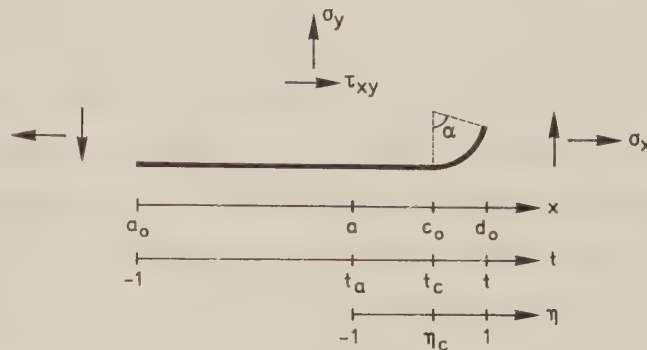


Fig. 5. Straight crack terminating in a circular arc.

The calculations were performed in two steps according to (15), so that the first solution covers the total interval $a_0 \leq x \leq d_0$ and the second solution covers the interval $a < x \leq d_0$ only, where a is an x -coordinate rather close to c_0 , see Fig. 5. For $x = a$ the parameter t is denoted t_a .

In the first solution, the dominating coefficient in the series expansion (2) is B_{k1} , corresponding to the analytical solution for a straight crack in the interval $a_0 \leq x \leq d_0$. The analytical expression in the case of a straight crack in the normalized interval $-1 \leq t \leq 1$ is

$$\begin{aligned} b_x(t) &= \frac{1}{\pi} \tau_{xy} T_1(t) (1-t^2)^{-1/2} \\ b_y(t) &= \frac{1}{\pi} \sigma_y T_1(t) (1-t^2)^{-1/2} \end{aligned} \quad (17)$$

and the first solution $b_k^{(1)}(t)$ in (15) is chosen to be the expressions (17).

In the second stage of solution over the interval $t_a < t \leq 1$, normalized to $-1 < \eta \leq 1$, an expression in the form (16) is assumed and the calculation is performed by the method of least squares.

In the calculations the following data were used: $t_a = 0.6$, $t_c = 0.9$, $N = 20$, $M = 4N$ and $Q = 50$. The solution is stable with respect to the displacement derivatives and only slight oscillations occur close to $\eta = -1$ even for α as large as $4\pi/9$.

6. The kinked crack

The kinked crack is straight between $x = a_0$ ($t = -1$) and $x = c_0$ ($t = t_c$) and provided with a straight kink between $x = c_0$ and $x = d_0$ ($t = +1$). The kink angle is α . The problem was treated in the same way as the smoothly curved crack in the previous section, i.e. by taking the first solution in (15) to be the analytical solution (17) for a straight crack, and then seek a second solution over an interval $t_a < t \leq 1$, covering the kink and a sufficiently large part of the main crack to ensure that the solution should be very close to the one given by (17) for $t \leq t_a$. The same data as for the smooth crack were used, i.e. $t_a = 0.6$, $t_c = 0.9$. Solutions were obtained from eq. (12) with $N = 20$ and $P = Q = 50$. In addition the method of least squares was used with $N = 20$, $M = 4N$ and $Q = 50$.

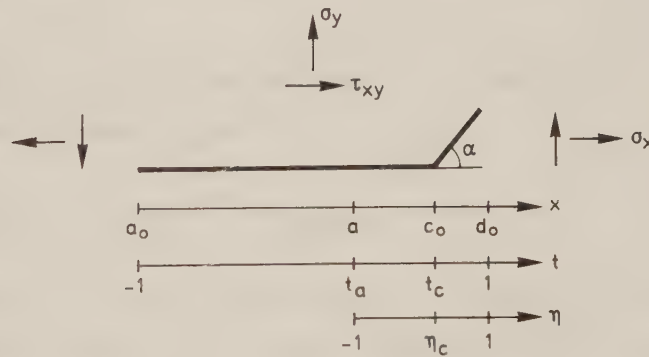


Fig. 6. Kinked crack.

The nature of the solution differs from the one obtained for the smoothly curved crack in the previous section in the same way as the solution in the case of a V-shaped crack differs from that of the arc-shaped one. Oscillations in the displacement derivatives develop as α increases and as judged from these oscillations, the accuracy of K_I and K_{II} is no better than about ten per cent. Use of the method of least squares does improve the accuracy somewhat, but not to be better than a few per cent.

7. Attempts to improve the solution

Since it was obvious that the singularity due to the sharp corner was responsible for the instability of the solution, an attempt was made to improve the accuracy by adding terms containing singularities of the appropriate power to the Chebyshev series in the second solution. Thus it was assumed that the second solution could be written in the form

$$\begin{aligned}
 c_k(\eta) = & (1-\eta^2)^{-1/2} \sum_{j=0}^N C_{kj} T_j(\eta) \\
 & + [D_{ks} r^{\lambda_1-1} + D_{ka} r^{\lambda_2-1}] U(\eta_c - \eta) \\
 & + [E_{ks} r^{\lambda_1-1} + E_{ka} r^{\lambda_2-1}] U(\eta - \eta_c) \quad -1 < \eta \leq 1 \quad (18)
 \end{aligned}$$

where $U(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

where $r = |\eta - \eta_c|$ and η_c corresponds to t_c , see Fig. 6. λ_1 and λ_2 are the first solutions to the two equations

$$\pm \lambda_{1,2} \sin[\alpha + \pi] + \sin[\lambda_{1,2}(\pi + \alpha)] = 0 \quad (19)$$

which result from a study of the equilibrium close to a notch with angle $(\pi - \alpha)$ [7]. Upper sign goes with first index. The lowest root of the upper equation, λ_1 , is associated with the symmetrical and the lowest root of the lower equation, λ_2 , with the antisymmetrical part of the displacement derivatives.

Condition (5) gives

$$\begin{aligned}
 C_{k0} = & -\frac{1}{\pi \lambda_1} [D_{ks} (1 + \eta_c)^{\lambda_1} + E_{ks} (1 - \eta_c)^{\lambda_1}] \\
 & -\frac{1}{\pi \lambda_2} [D_{ka} (1 + \eta_c)^{\lambda_2} + E_{ka} (1 - \eta_c)^{\lambda_2}] \quad (20)
 \end{aligned}$$

This approach actually proved to increase the stability somewhat, but still the accuracy of K_I and K_{II} did not reach the one per cent level, judging from the remaining oscillations.

To eliminate the risk that the part of $c_k(\eta)$ that contains Chebyshev polynomials contributes to the instability because of the inclusion of a square root singularity at the left (non-singular) end of the interval ($\eta = -1$), an attempt was made to improve the stability by exchanging the series of Chebyshev polynomials by a series containing Jacobi polynomials $P_j^{(-0.5, 0.5)}$. The associated weight-function is non-singular at $\eta = -1$ and square root singular at $\eta = +1$. Thus, instead of (18) the following assumption was made:

$$\begin{aligned}
 c_k(\eta) = & (1+\eta)^{1/2}(1-\eta)^{-1/2} \sum_{j=0}^N C_{kj} P_j^{(-0.5, 0.5)}(\eta) \\
 & + [D_{k\beta} r^{\lambda_1-1} + D_{k\alpha} r^{\lambda_2-1}] U(\eta - \eta_c) \\
 & + [E_{k\beta} r^{\lambda_1-1} + E_{k\alpha} r^{\lambda_2-1}] U(\eta - \eta_c) \quad -1 \leq \eta \leq 1
 \end{aligned} \tag{21}$$

Condition (5) gives C_{k0} as in (20).

The stability of the solution did not improve significantly and K_I and K_{II} could not be calculated within an accuracy of one percent, at least not with a reasonable numerical effort.

8. Conclusions

By use of finite element methods an accuracy of about one per cent or slightly better is readily obtained for the type of problems studied. The conclusion therefore seems to be that the method with singular integral equations solved by using Chebyshev quadratures is superior to the finite element method for smoothly curved cracks, but inferior for kinked cracks. Some improvement of the integral equation methods is possible, for instance by solving equations in the sense of the method of least squares or by

obtaining a solution in two steps, the second one designated to improve the accuracy in the neighbourhood of the kink. Nevertheless, the accuracy seems to be dramatically decreased when proceeding from smoothly curved cracks to cracks with kinks.

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